TORIC FANO MANIFOLDS WITH NEF TANGENT BUNDLES

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Abstract. In this note we prove that any toric Fano manifold with nef tangent bundle is a product of projective spaces. In particular, it implies that Campana-Peternell conjecture hold for toric manifolds.

1. Notation and main result

We will use standard notation for polytopes and toric varieties, as it can be found in [CLS],[Fu],[Od].

Let $N \cong \mathbb{Z}^d$ be a d-dimensional lattice and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^d$ the dual lattice with $\langle , \rangle$ the nondegenerate pairing. As usual, $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^d$ and $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^d$ (respectively $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$) will denote the rational (respectively real) scalar extensions.

A subset $P \subseteq M_{\mathbb{R}}$ is called a polytope if it is the convex hull of finitely many points in $M_{\mathbb{R}}$. The face of P is denoted by $F \preceq P$. The set of vertices and facets of P are denoted by $\mathscr{V}(P)$ and $\mathscr{F}(P)$ respectively. If $\mathscr{V}(P) \subseteq M_{\mathbb{Q}}$ (respectively $\mathscr{V}(P) \subseteq M$) then P is called a rational polytope (respectively a lattice polytope).

If P is a rational polytope with $0 \in \mathrm{int}P$, the dual polytope of P is defined by

$$P^* := \{y \in N_{\mathbb{R}} | \langle x, y\rangle \geq -1, \forall x \in P\},$$

which is also a rational polytope with $0 \in \mathrm{int}P^*$. The fan $\mathscr{N}_P := \{\mathrm{pos}(F) : F \preceq P^*\}$ is called the normal fan of P. Here $\mathrm{pos}(F)$ denotes the cone positively generated by the face F (also called positive hull of F). It is well-known that a fan Σ in $N_{\mathbb{R}}$ defines a toric variety $X_\Sigma := X(N, \Sigma)$, which automatically admits a torus action and has a Zariski open and dense orbit:

$$T_N \times X_\Sigma \to X_\Sigma,$$

where $T_N \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$. We denote $X_P := X_{\mathscr{N}_P}$ the toric variety associated with the normal fan $\mathscr{N}_P$ of the polytope P. It is known that X_P is nonsingular if and only if the vertices of any facet of P^* form a $\mathbb{Z}$-basis of the lattice M.

A d-dimensional polytope $P \subseteq M_{\mathbb{R}}$ with $0 \in \mathrm{int}P$ is called reflexive polytope if both P and P^* are lattice polytopes. A complex variety X is called a Gorenstein Fano variety if X is projective, normal and its anticanonical divisor is an ample Cartier divisor. The following theorem (see [Ni1]) classifies Gorenstein toric Fano varieties by reflexive polytopes:

Theorem 1.1. *Under the map $P \longmapsto X_P$ reflexive polytopes correspond uniquely up to isomorphism to Gorenstein toric Fano varieties. There are only finitely many isomorphism types of d-dimensional reflexive polytopes.*

A Cartier divisor D on a nonsingular variety X is called a nef divisor if the intersectional number $D \cdot C \geq 0$ for any irreducible curve $C \subset X$. A line bundle L is called a nef line bundle if the associated Cartier divisor (i.e., $L = \mathscr{O}_X(D)$) is a nef divisor. A vector bundle E over X is called a nef vector bundle if the tautological

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line bundle $\mathscr{O}_{\mathbb{P}(E^*)}(1)$ on the projective bundle $P(E^*)$ is a nef line bundle. In [CP], Campana and Peternell conjectured that any Fano manifold with nef tangent bundle is a rational homogeneous manifold. In this note we confirm this conjecture for toric Fano manifold, in fact we get more and obtain the following main theorem:

Theorem 1.2. *Any toric Fano manifold with nef tangent bundle is a product of projective spaces.*

2. Cartier divisors on complete toric varieties

A fan Σ in $N_{\mathbb{R}}$ is complete iff its support $|\Sigma| := \cup_{\sigma\in\Sigma}\sigma$ is the whole space $N_{\mathbb{R}}$, which is also equivalent to that the associated toric variety $X(N,\Sigma)$ is compact in classical topology ([CLS, Theorem 3.1.9]).

Let $\Sigma(k)$ denote the set of k-dimensional cones of the complete fan Σ. The elements in $\Sigma(1)$ are called rays, and given $\tau \in \Sigma(1)$, let u_τ denote the unique minimal generator of $N\cap\tau$. By orbit-cone correspondence ([CLS, Theorem 3.2.6]), a ray $\tau\in\Sigma(1)$ gives a T_N-invariant Cartier divisor D_τ. On a complete toric variety we may write any Cartier divisor as a linear combination of T_N-invariant Cartier divisors. Let $D=\sum_{\tau\in\Sigma(1)} a_\tau D_\tau$ be a Cartier divisor on a complete toric variety X_Σ, its support function $\phi_D : N_{\mathbb{R}}\to\mathbb{R}$ is determined by the following properties:

(1) ϕ_D is linear on each cone $\sigma\in\Sigma$.
(2) $\phi_D(u_\tau) = -a_\tau$.
(3) For each cone $\sigma\in\Sigma$ there is a $m_\sigma\in M$ such that $\phi_D(u)=\langle m_\sigma, u\rangle$ for all $u\in\sigma$ and $\langle m_\sigma, u_\tau\rangle = -a_\tau$ for all $\tau\in\sigma(1)$.

Proposition 2.1. ([CLS, Theorem 6.1.10 and Theorem 6.2.12]) *Let $D=\sum_{\tau\in\Sigma(1)} a_\tau D_\tau$ be a Cartier divisor on a complete toric variety X_Σ and denote*

$$P_D = \{m\in M_{\mathbb{R}} | \langle m, u_\tau\rangle \geq -a_\tau, \forall\tau\in\Sigma(1)\}.$$

Then the following are equivalent:

(1) D is basepoint free.
(2) D is a nef divisor.
(3) ϕ_D is a upper convex function.
(4) $m_\sigma\in P_D$ for all $\sigma\in\Sigma(d)$.
(5) $\phi_D(u)=\min_{m\in P_D}\langle m,u\rangle$ for all $u\in N_{\mathbb{R}}$.

The support function ϕ_D of a Cartier divisor D on a complete toric variety X_Σ is called *strictly convex* if it is upper convex and for each $\sigma\in\Sigma(d)$ satisfies

$$\langle m_\sigma, u\rangle = \phi_D(u) \Longleftrightarrow u\in\sigma.$$

Proposition 2.2. ([CLS, Theorem 6.1.15 and Corollary 6.1.16])*A Cartier divisor D on a complete toric variety X_Σ is ample if and only if its support function ϕ_D is strictly convex. If D is ample then P_D is a full dimensional lattice polytope whose normal fan is Σ.*

3. Complete toric variety with reductive automorphism group

The automorphism group $\mathrm{Aut(X_\Sigma)}$ of a nonsingular complete toric variety X_Σ was firstly studied by Demazure in [De, Section 4]. Identifying the elements of the Lie algebra of $\mathrm{Aut(X_\Sigma)}$ with the invariant differential operators on the coordinate ring of X_Σ, Demazure gave a very simple description of the structure of the Lie algebra of $\mathrm{Aut(X_\Sigma)}$ using the Demazure root system named after him. The Demazure root system $\mathscr{R}$ of $\mathrm{Aut(X_\Sigma)}$ has a very simple description:

$$\mathscr{R} = \{m\in M | \exists\tau\in\Sigma(1) : \langle u_\tau, m\rangle = -1, \langle u_{\tau'}, m\rangle \geq 0, \forall\tau'\in\Sigma(1)\backslash\{\tau\}\}.$$

Note here we use notation of [Ni1, Ni2], which are different form those in [De] by a minus signature. The Demazure roots in $\mathscr{R}\cap-\mathscr{R}=\{m\in\mathscr{R}| -m\in\mathscr{R}\}$ are called

semisimple roots. $\mathrm{Aut(X_\Sigma)}$ is a reductive algebraic group iff all Demazure roots in $\mathscr{R}$ are semisimple, i.e., $\mathscr{R} = -\mathscr{R} := \{-m|m \in \mathscr{R}\}$. The following proposition of Nill, Benjamin's will be used in proving our main theorem in the next section.

Proposition 3.1. [Ni2, Proposition 3.18] *A d-dimensional complete toric variety is isomorphic to a product of projective spaces iff there are d-linearly independent semisimple roots.*

4. Toric Fano manifolds with nef tangent bundles

The projective space $\mathbb{P}^n$ is a toric Fano manifold and the following exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^d} \longrightarrow \mathscr{O}_{\mathbb{P}^n}(1)^{d+1} \longrightarrow T_{\mathbb{P}^d} \longrightarrow 0$$

is called the Euler sequence of $\mathbb{P}^d$. The following theorem is a toric generalization of this result.

Theorem 4.1. ([CLS, Theorem 8.1.6]) *Let X_Σ be a toric manifold associated with the complete fan Σ, then we have the following generalized Euler sequence*

$$0 \longrightarrow \mathscr{O}_{X_\Sigma}^{\oplus\rho} \longrightarrow \oplus_{\tau\in\Sigma(1)}\mathscr{O}_{X_\Sigma}(D_\tau) \longrightarrow T_{X_\Sigma} \longrightarrow 0,$$

where ρ is the Picard number of X_Σ.

In particular, the canonical divisor and anticanonical divisor of the toric manifold X_Σ are respectively given by

$$K = -\sum_{\tau\in\Sigma(1)} D_\tau; \qquad K^* = \sum_{\tau\in\Sigma(1)} D_\tau.$$

Proposition 4.2. *Assume X_Σ is a toric manifold with nef tangent bundle, then for any $\tau \in \Sigma(1)$, the associated T_N-invariant Cartier divisor D_τ is a nef divisor.*

Proof. Since D_τ is T_N-invariant, it is a smooth hypersurface locating inside X_Σ, we have the following exact sequnce

$$0 \longrightarrow T_{D_\tau} \longrightarrow T_{X_\Sigma}\Big|_{D_\tau} \longrightarrow N_{D_\tau} \longrightarrow 0,$$

Note the normal sheaf of D_τ in X_Σ could be identified with $\mathscr{O}_{X_\Sigma}(D_\tau)$. Since the tangent bundle T_{X_Σ} is a nef vector bundle, so is the quotient bundle N_{D_τ} by [DPS, Proposition 1.15]. Hence D_τ is a nef divisor. $\square$

Theorem 4.3. *The tangent bundle of a toric manifold with nef tangent bundle is Griffiths semipositive.*

Proof. Since $\mathscr{O}_{X_\Sigma}(D_\tau)$ is a nef line bundle, D_τ is basepoint free by Proposition 2.1. Hence $\mathscr{O}_{X_\Sigma}(D_\tau)$ is a semipositive line bundle. Hence the direct sum bundle $\oplus_{\tau\in\Sigma(1)}\mathscr{O}_{X_\Sigma}(D_\tau)$ is a Griffiths semipositve. The Griffiths semipositivity of tangent bundle T_{X_Σ} follows by the generalized Euler sequence via using [Ya, Proposition 3.5]. $\square$

Note Theorem 4.3 already implies that Campana-Peternell conjecture holds for toric fano manifolds. In fact, from Theorem 4.3 we know a toric fano manifold X_Σ with nef tangent bundle has nonnegative holomorphic bisectional curvature and positive Ricci curvature. By Mok's theorem [Mo], X_Σ is biholomorphic to the product of Hermitian symmetric manifolds. In particular, $\mathrm{Aut(X_\Sigma)}$ is a reductive algebraic group. In the rest part of this note we will give a more precise structure description of a toric fano manifold with nef tangent bundle.

Let ϕ_{D_τ} be the support function associated with T_N-invariant cartier divisor D_τ. Then on each open cone $\sigma \in \Sigma(d)$,

$$\phi_{D_\tau}(u) = \langle m_\sigma, u\rangle,\ \forall u \in \sigma,$$

for some $m_\sigma \in M$.

Proposition 4.4. *If the cartier divisor D_τ is nef then $\{m_\sigma|\sigma \in \Sigma(d)\}$ are semisimple Demazure roots of* $\mathrm{Aut(X_\Sigma)}$.

Proof. By the definition of the data m_σ for the Cartier divisor D_τ, we have $\phi_{D_\tau}(u_\tau) = \langle m_\sigma, u_\tau\rangle = -1$. Now fix a $\sigma \in \Sigma(d)$. By (4) of Proposition 2.1, $m_\sigma \in P_{D_\tau}$ if D_τ is basepoint free. Note now

$$P_{D_\tau} = \{m \in M|\langle m, u_\tau\rangle \geq -1 \text{ and } \langle m, u_{\tau'}\rangle \geq 0, \forall \tau' \in \Sigma(1)\backslash\{\tau\}\},$$

hence we have

$$\langle m_\sigma, u_{\tau'}\rangle \geq 0, \forall \tau' \in \Sigma(1)\backslash\{\tau\}\},$$

therefore m_σ is a Demazure root of $\mathrm{Aut(X_\Sigma)}$. Since $\mathrm{Aut(X_\Sigma)}$ is reductive, it is also a semisimple root. $\square$

Proposition 4.5. *If X_Σ is a toric Fano manifold with nef tangent bundle then* $\mathrm{Aut(X_\Sigma)}$ *has d linearly independent semisimple roots.*

Proof. Let $\tau_1, \cdots, \tau_m$ denote all of 1-dimensinal cones of Σ and $u_{\tau_1}, \cdots, u_{\tau_m}$ their primitive generating vectors, and $D_1, \cdots, D_m$ the corresponding T_N-invariant basepoint free Cartier divisors. The support function ϕ_{D_i} of D_i satisfies that

$$\phi_{D_i}(x) = \langle m_\sigma, x\rangle, \quad \forall x \in \sigma \in \Sigma(d),$$

where $\langle m_\sigma, u_{\tau_i}\rangle = -1$ and $\langle m_\sigma, u_{\tau_j}\rangle \geq 0$ for $j \neq i$. Let $\{m_\sigma^i\}$ be the set of semisimple Demazure roots associated with the Cartier divisor D_i. Since each D_i is a basepoint free divisor, the associated polytope

$$P_{D_i} = \{m \in M_{\mathbb{R}}|\langle m, u_{\tau_i}\rangle \geq -1 \text{ and } \langle m, u_{\tau_j}\rangle \geq 0\ \forall\ j \neq i\}$$

is a convex polytope. Note for $i \neq j$,

$$P_{D_i} \cap P_{D_j} = \{m \in M_{\mathbb{R}}|\langle m, u_{\tau_j}\rangle \geq 0 \text{ for } j = 1, \cdots, m\} = \mathrm{pos}(\tau_1, \cdots, \tau_m)^\vee$$

is $\{0\}$, since Σ is complete the convex cone $\mathrm{pos}(\tau_1, \cdots, \tau_m) = N_{\mathbb{R}}$.

The anticanonical divisor of X_Σ is given by $K^* = D_1 + \cdots + D_m$ and it is an ample divisor, the associated polytope

$$P_{K^*} = \{m \in M_{\mathbb{R}}|\langle m, u_{\tau_i}\rangle \geq -1 \text{ for } i = 1, \cdots, m\}$$

is a full dimensional polytope by Proposition 2.2. Note that

$$P_{K^*} = P_{D_1} \cup \cdots \cup P_{D_m}.$$

Since $0 \neq m_\sigma^i \in P_{D_i}$, we have for any $\sigma, \sigma' \in \Sigma(d)$ that $m_\sigma^i \neq m_{\sigma'}^j$ if $i \neq j$. Note $\{m_\sigma^i\}$ are vertices of P_{D_i}, however none of them are vertices of P_{K^*} though $\{m_\sigma^i|\sigma \in \Sigma(d)\} \subset P_{K^*}$. In fact m_σ^i can't lie in the intersection of two facets of P_{K^*}, hence it is not inside the facets with codimension ≥ 2 of P_{K^*}. But each m_σ^i is in the codimensional one facet $H_i = \{x \in M|\langle m, u_{\tau_i}\rangle = -1\}$ of P_{K^*}, and for $i \neq j$, the points $\{m_\sigma^i|\sigma \in \Sigma(d)\}$ and $\{m_\sigma^j|\sigma \in \Sigma(d)\}$ locate in the different facets of P_{K^*}.

Now let v be any vertex of P_{K^*} which has at least d codimensional one facets of P_{K^*}, assume $H_{i_1}, \cdots, H_{i_k}(k \geq d)$ are those facets passing through the vertex v and $H_{i_1} \cap \cdots \cap H_{i_k} = \{v\}$. Now fix a cone $\sigma \in \Sigma(d)$, since $\{m \in M|\langle m, u_{\tau_{i_j}}\rangle \geq -1, j = 1, \cdots, k\}$ is a d-dimensional cone $\mathrm{Cone}(P_{K^*} \cap M_{\mathbb{R}} - v)$, the vectors $m_\sigma^{i_1} - v, \cdots, m_\sigma^{i_k} - v$ form a basis of $N_{\mathbb{R}}$. Since $m_\sigma^{i_1}, \cdots, m_\sigma^{i_k}$ are on the different facets of cone $\mathrm{Cone}(P_{K^*} \cap M_{\mathbb{R}} - v)$, without loss of generality we may assume $m_\sigma^{i_1} - v, \cdots, m_\sigma^{i_d} - v$are linearly independent. Then after a translation, $m_\sigma^{i_1}, \cdots, m_\sigma^{i_d}$

are still linearly independent. By Proposition 4.4,$m_\sigma^{i_1}, \cdots, m_\sigma^{i_d}$ are semisimple Demazure roots of $\mathrm{Aut}(X_\Sigma)$, hence it has d linearly independent semisimple roots. $\square$

Now our main result Theorem 1.2 follows from Proposition 3.1 and Proposition 4.5.

References

[CP] F. Campana and T. Peternell, Projective manifolds whose tangent bundles are numerically effective. *Math. Ann.* **289** (1991), 169–187.

[CLS] David A. Cox, John B. Little and Henry K. Schenck, *Toric varieties,* Graduate Studies in Mathematics, Vol **124**, American Mathematical Society, Providence, RI, 2011.

[DPS] J.-P. Demailly, T. Peternell and M. Schneider, Compact complex manifolds with numerically effective tangent bundles. *J. Algebraic Geom.* **3** (1994), 295–345.

[De] M. Demazure, Sous-groupes algébriques de rang maximum du groupe de Cremona. (French) *Ann. Sci. École Norm. Sup.* **3** (1970) 507–588.

[Fu] W. Fulton, *Introduction to toric varieties,* Annals of Mathematics Studies, **131**, The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[Mo] N. Mok, The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature. *J. Differential Geom.* **27** (1988), 179–214.

[Ni1] B. Nill, Gorenstein toric Fano varieties. *Manuscripta Math.* **116** (2005), 183–210.

[Ni2] B. Nill, Complete toric varieties with reductive automorphism group. *Math. Z.* **252** (2006), 767–786.

[Od] T. Oda, Convex bodies and algebraic geometry. An introduction to the theory of toric varieties. Translated from the Japanese. Ergebnisse der Mathematik und ihrer Grenzgebiete **15.** Springer-Verlag, Berlin, 1988.

[Ya] Q.-L. Yang, (k, s)-positivity and vanishing theorems for compact Kähler manifolds. *Internat. J. Math.* **22** (2011), 545–576.

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